

# A central limit type theorem for Gaussian mixture approximations to the nonlinear filtering problem

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## Abstract

Approximating the solution of the nonlinear filtering problem with Gaussian mixtures has been a very popular method since the 1970s. However, the vast majority of such approximations are introduced in an ad-hoc manner without theoretical grounding. This work is a continuation of [4, 5], where we described a rigorous Gaussian mixture approximation to the solution of the filtering problem. We deduce here a refined estimate of the rate of convergence of the approximation. We do this by proving a central limit type theorem for the error process. We also find the optimal variances of the Gaussian measures are of order  $1/\sqrt{n}$ . This implies, in particular, that the mean square error of the approximation as defined in [4, 5] is of order  $1/n$ .

## 1 Introduction

The stochastic filtering problem deals with the estimation of an evolving dynamical system, called the *signal*, based on *partial observations* and a priori stochastic model. The signal is modelled by a stochastic process denoted by  $X = \{X_t, t \geq 0\}$ , defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The signal process is not available to observe directly; instead, a partial observation is obtained and it is modelled by a process  $Y = \{Y_t, t \geq 0\}$ . The information available from the observation up to time  $t$  is defined as the filtration  $\mathcal{Y} = \{\mathcal{Y}_t, t \geq 0\}$  generated by the observation process  $Y$ . In this setting, we want to compute  $\pi_t$  — the conditional distribution of  $X_t$  given  $\mathcal{Y}_t$ .

The description of a numerical approximation for  $\pi_t$  should contain the following three parts: the class of approximations; the law of evolution of the approximation; and the method of measuring the approximating error. Gaussian mixtures approximations are numerical schemes that approximate  $\pi_t$  with random measures of the form

$$\sum_j a_j(t) \Gamma_{v_j(t), \omega_j(t)},$$

where  $a_j(t)$  is the weight of the Gaussian (generalised) particle,  $\Gamma_{v_j(t), \omega_j(t)}$  is the Gaussian measure with mean  $v_j(t)$  and covariance matrix  $\omega_j(t)$ . The evolution of the weights, the mean and the covariance matrices satisfy certain stochastic differential equations which are numerically solvable.

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Studies of Gaussian mixtures approximations in the context of Bayesian estimation have been developing for nearly fifty years since 1970s (see, for example, [5] for a survey of the existing work). However, not until recently can we see a theoretical analysis and  $L^2$ -convergence rate for such approximating system obtained by Crisan and Li ([4, 5]). In addition to the  $L^2$ -convergence, it is also of great importance that one can recalibrate the error of the approximation and characterise its exact convergence rate, in other words, prove a central limit theorem type result of such approximation.

Various other approximations to the nonlinear filtering problems have been shown to satisfy central limit type theorems. Del Moral, Guionnet, and Miclo (see [10], [11], [12]) deduced central limit type results (CLT) for unweighted particle filters using the interacting particle systems. Crisan and Xiong ([7]) proved a CLT result for the classical nonlinear filtering case and obtained the rate as  $n^{(1-\alpha)/2}$  for any  $\alpha > 0$ ; and this result was later improved by Xiong and Zeng ([25]) up to  $n^{1/2}$ . Similar CLT results were also obtained for the discrete time filtering framework by Chopin ([2]) and Kunsch ([16]).

However, to the authors' knowledge, there has been no theoretical analysis of the convergence in distribution for the Gaussian mixture approximations to the filtering problem, and no corresponding central limit type result was proven for this type of approximations. The main purpose of this paper is to fill this gap and obtain a CLT result for such approximation.

## 1.1 Contribution of the paper

This paper is a continuation of the work done in [5]. In particular, we deduce here a central limit theorem for the algorithm presented in [4, 5]. To be specific, let  $\pi = \{\pi_t; t \geq 0\}$  be the conditional distribution and  $\pi^{n,\varepsilon} = \{\pi_t^{n,\varepsilon}; t \geq 0\}$  be the approximation of the conditional distribution constructed in [5] (and in Section 3 in this paper) using mixtures of Gaussian measures, where  $n$  is the number of Gaussian measures and  $\varepsilon$  is a positive parameter measuring the amount of ‘‘Gaussianity’’ (see discussion after (3.3) for details). We obtain a central limit type result and show that the recalibrated error converges in distribution to a unique measure-valued process as  $n$  increases; in addition, we find the optimal value for  $\varepsilon$ .

To do this we introduce the following measure-valued processes  $\bar{U}^{n,\varepsilon} = \{\bar{U}_t^{n,\varepsilon}; t \geq 0\}$  and  $U^{n,\varepsilon} = \{U_t^{n,\varepsilon}; t \geq 0\}$  as

$$\bar{U}_t^{n,\varepsilon} = n^\varepsilon (\pi_t^{n,\varepsilon} - \pi_t) \quad \text{and} \quad U_t^{n,\varepsilon} = n^\varepsilon (\rho_t^{n,\varepsilon} - \rho_t),$$

where  $\rho$  ( $\rho^{n,\varepsilon}$ ) is the unnormalised version of  $\pi$  ( $\pi^{n,\varepsilon}$ ) (see Section 3 for details). Then we have the following.

**Theorem 1.1.** *The  $L^2$ -convergence rate of  $\pi^{n,\varepsilon}$  ( $\rho^{n,\varepsilon}$ ) to  $\pi$  ( $\rho$ ) is  $(\frac{1}{n})^{\min\{2\varepsilon, 1\}}$  for  $\varepsilon > 0$ . When  $0 < \varepsilon \leq 1/2$ , for each  $\varepsilon$ , there is a unique measure-valued process  $U^\varepsilon = \{U_t^\varepsilon; t \geq 0\}$  solving the following stochastic PDE, given any test function in  $\varphi \in C_b^6(\mathbb{R}^d)$ :*

$$U_t^\varepsilon(\varphi) = U_0^\varepsilon(\varphi) + \int_0^t U_s^\varepsilon(A\varphi)ds + \int_0^t U_s^\varepsilon(h\varphi)dY_s + \Lambda_t^{\varphi,\varepsilon},$$

where the definitions of the operator  $A$ , the function  $h$  and  $\Lambda^\varphi$  can be found in subsequent sections; and  $U^{n,\varepsilon}$  forms a tight sequence and converges in distribution to the process  $U^\varepsilon$ . In addition,  $\bar{U}^{n,\varepsilon}$  converges in distribution to a measure-valued process  $\bar{U}^\varepsilon = \{\bar{U}_t^\varepsilon; t \geq 0\}$ , which is defined by

$$\bar{U}_t^\varepsilon(\varphi) = \frac{1}{\rho_t(\mathbf{1})} (U_t^\varepsilon(\varphi) - \pi_t(\varphi)U_t^\varepsilon(\mathbf{1})).$$

When  $\varepsilon > 1/2$ , the process  $\{U^{n,\varepsilon}\}_n$  ( $\{\bar{U}^{n,\varepsilon}\}_n$ ) is divergent. In other words, the central limit theorem is obtained when  $\varepsilon \in (0, 1/2]$ , and among this range  $\varepsilon = 1/2$  gives the optimal  $L^2$ -convergence rate.

**Remark 1.2.** The proof of the  $L^2$ -convergence rate of  $\pi^{n,\varepsilon}(\rho^{n,\varepsilon})$  to  $\pi(\rho)$  can be found in Section 4 of [5], hence we will not prove this part of Theorem 1.1 in this paper.

The following is a summary of the contents of the paper.

In Section 2, we review the key results of stochastic filtering theory. The filtering framework is introduced first, with the focus on the problems where the signal  $X$  and observation  $Y$  are diffusion processes and the filtering equations are presented.

Section 3 contains the description of the generalised particle filters with Gaussian mixtures. These approximations use mixtures of Gaussian measures which will be set out, with the aim of estimating the solutions to the Zakai and the Kushner-Stratonovich equations. The Multinomial branching algorithm is chosen to be the associated correction mechanism.

Sections 4 and 5 contain the main result of the paper, which is the central limit theorem associated to the approximating system. The analysis is proceeded in a standard manner. In Section 4, based on the evolution equations of the approximating systems derived in [5], the error between the Gaussian mixture approximation and the true solution is recalibrated and shown to be a tight sequence. In section 5, we find its limit in distribution and show this limiting process is unique.

This paper is concluded in Section 6 and with an Appendix which contains some additional results required in the main body of the paper.

## 1.2 Notations

- $\mathbb{R}^d$  - the  $d$ -dimensional Euclidean space.
- $\bar{\mathbb{R}}^d$  - the one-point compactification of  $\mathbb{R}^d$ .
- $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  - the state space of the signal.  $\mathcal{B}(\mathbb{R}^d)$  is the associated Borel  $\sigma$ -algebra.
- $B(\mathbb{R}^d)$  - the space of bounded  $\mathcal{B}(\mathbb{R}^d)$ -measurable functions from  $\mathbb{R}^d$  to  $\mathbb{R}$ .
- $\mathcal{P}(\mathbb{R}^d)$  - the family of Borel probability measures on space  $\mathbb{R}^d$ .
- $C_b(\mathbb{R}^d)$  - the space of bounded continuous functions on  $\mathbb{R}^d$ .
- $C_b^m(\mathbb{R}^d)$  - the space of bounded continuous functions on  $\mathbb{R}^d$  with bounded derivatives to order  $m$ .
- $C_0^m(\mathbb{R}^d)$  - the space of continuous functions on  $\mathbb{R}^d$ , vanishing at infinity with continuous partial derivatives up to order  $m$ .
- $\|\cdot\|$  - the Euclidean norm for a  $d \times p$  matrix  $a$ ,  $\|a\| = \sqrt{\sum_{i=1}^d \sum_{j=1}^p a_{ij}^2}$ .
- $\|\cdot\|_\infty$  - the supremum norm for  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ :  $\|\varphi\|_\infty = \sup_{x \in \mathbb{R}^d} \|\varphi(x)\|$ .
- $\|\cdot\|_{m,\infty}$  - the norm such that for  $\varphi$  on  $\mathbb{R}^d$ ,  $\|\varphi\|_{m,\infty} = \sum_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^d} |D_\alpha \varphi(x)|$ , where  $\alpha = (\alpha^1, \dots, \alpha^d)$  is a multi-index and  $D_\alpha = (\partial_1)^{\alpha_1} \dots (\partial_d)^{\alpha_d}$ .
- $\mathcal{M}_F(\mathbb{R}^d)$  - the set of finite measures on  $\mathbb{R}^d$ .
- $\mathcal{M}_F(\bar{\mathbb{R}}^d)$  - the set of finite measures on  $\bar{\mathbb{R}}^d$ .
- $D_{\mathcal{M}_F(\mathbb{R}^d)}[0, T]$  - the space of càdlàg functions (or right continuous functions with left limits)  $f : [0, T] \rightarrow \mathcal{M}_F(\mathbb{R}^d)$ .
- $D_{\mathcal{M}_F(\mathbb{R}^d)}[0, \infty)$  - the space of càdlàg functions (or right continuous functions with left limits)  $f : [0, \infty) \rightarrow \mathcal{M}_F(\mathbb{R}^d)$ .

## 2 The Filtering Problem and Key Result

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space together with a filtration  $(\mathcal{F}_t)_{t \geq 0}$  which satisfies the usual conditions. On  $(\Omega, \mathcal{F}, \mathbb{P})$  we consider an  $\mathcal{F}_t$ -adapted process  $X = \{X_t; t \geq 0\}$  taking values on  $\mathbb{R}^d$ . To be specific, let  $X = (X^i)_{i=1}^d$  be the solution of a  $d$ -dimensional stochastic differential equation driven by a  $p$ -dimensional Brownian motion  $V = (V^j)_{j=1}^p$ :

$$X_t^i = X_0^i + \int_0^t f^i(X_s) ds + \sum_{j=1}^p \int_0^t \sigma^{ij}(X_s) dV_s^j, \quad i = 1, \dots, d. \quad (2.1)$$

We assume that both  $f = (f^i)_{i=1}^d : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma = (\sigma^{ij})_{i=1, \dots, d; j=1, \dots, p} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times p}$  are globally Lipschitz. Under the globally Lipschitz condition, (2.1) has a unique solution (e.g., Theorem 5.2.9 in [15]).

Let  $h = (h_i)_{i=1}^m : \mathbb{R}^d \rightarrow \mathbb{R}^m$  be a bounded measurable function. Let  $W$  be a standard  $\mathcal{F}_t$ -adapted  $m$ -dimensional Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$  independent of  $X$ , and  $Y$  be the process which satisfies the following evolution equation

$$Y_t = Y_0 + \int_0^t h(X_s) ds + W_t, \quad (2.2)$$

This process  $Y = \{Y_t; t \geq 0\}$  is called the *observation* process. Let  $\{\mathcal{Y}_t, t \geq 0\}$  be the usual augmentation of the filtration associated with the process  $Y$ , viz  $\mathcal{Y}_t = \sigma(Y_s, s \in [0, t]) \vee \mathcal{N}$ .

As stated in the introduction, the filtering problem consists in determining the conditional distribution  $\pi_t$  of the signal  $X$  at time  $t$  given the information accumulated from observing  $Y$  in the interval  $[0, t]$ ; that is, for  $\varphi \in B(\mathbb{R}^d)$ ,

$$\pi_t(\varphi) \triangleq \int_{\mathbb{R}^d} \varphi(x) \pi_t(dx) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t]. \quad (2.3)$$

Throughout this paper we make the following assumption.

**Assumption (A).** Assume that the coefficients  $f^i$  and  $\sigma^{ij}$  are bounded and six times differentiable, and  $h^i$  is twice differentiable and has bounded derivatives. That is,  $f^i, \sigma^{ij} \in C_b^6(\mathbb{R}^d)$  and  $h^i \in C_b^2(\mathbb{R}^d)$ .

Let  $\tilde{\mathbb{P}}$  be a new probability measure on  $\Omega$ , under which the process  $Y$  is a Brownian motion. To be specific, let  $Z = \{Z_t, t \geq 0\}$  be the process defined by

$$Z_t = \exp \left( - \sum_{i=1}^m \int_0^t h^i(X_s) dW_s^i - \frac{1}{2} \sum_{i=1}^m \int_0^t h^i(X_s)^2 ds \right), \quad t \geq 0; \quad (2.4)$$

and we introduce a probability measure  $\tilde{\mathbb{P}}^t$  on  $\mathcal{F}_t$  by specifying its Radon-Nikodym derivative with respect to  $\mathbb{P}$  to be given by  $Z_t$ . We finally define a probability measure  $\tilde{\mathbb{P}}$  which is equivalent to  $\mathbb{P}$  on  $\bigcup_{0 \leq t < \infty} \mathcal{F}_t$ . Then we have the following Kallianpur-Striebel formula (see [14])

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(\mathbf{1})} \quad \tilde{\mathbb{P}}(\mathbb{P}) - a.s. \quad \text{for } \varphi \in B(\mathbb{R}^d), \quad (2.5)$$

where  $\rho_t$  is an  $\mathcal{Y}_t$ -adapted measure-valued process satisfying the following **Zakai Equation** (see [26]).

$$\rho_t(\varphi) = \pi_0(\varphi) + \int_0^t \rho_s(A\varphi) ds + \int_0^t \rho_s(\varphi h^\top) dY_s, \quad \tilde{\mathbb{P}} - a.s. \quad \forall t \geq 0 \quad (2.6)$$

for any  $\varphi \in \mathcal{D}(A)$ . In (2.6), operator  $A$  is the infinitesimal generator associated with the signal process  $X$

$$A = \sum_{i=1}^d f^i \frac{\partial}{\partial x^i} + \sum_{i=1}^d \sum_{j=1}^d a^{ij} \frac{\partial^2}{\partial x^i \partial x^j}, \quad (2.7)$$

where  $a = (a^{ij})_{i,j=1,\dots,d} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$  is the matrix-valued function defined as  $a = \frac{1}{2} \sigma^\top \sigma$ ; and  $\mathcal{D}(A)$  is the domain of  $A$ .

Also the process  $\rho = \{\rho_t; t \geq 0\}$  is called the unnormalised conditional distribution of the signal.

In the following we will obtain the central limit theorem for the associated generalised particle filters with Gaussian mixtures. We denote by  $\pi^{n,\varepsilon} = \{\pi_t^{n,\varepsilon}; t \geq 0\}$  the approximating measures of the solution of the filtering problem, where  $n$  is the number of Gaussian measures in the approximating system, and  $\varepsilon$  is a parameter measuring the amount of ‘‘Gaussianity’’ of the generalised particles.

### 3 Gaussian Mixtures Approximation

For ease of notations, we assume, hereinafter from this section, that the state space of the signal is one-dimensional. For clarity we describe the Gaussian mixture approximation introduced in [4, 5] below in this section. All the results presented here can be extended without significant technical difficulties to the multi-dimensional case.

Firstly, we let  $\Delta = \{0 = \delta_0 < \delta_1 < \dots < \delta_N = T\}$  be an equidistant partition of the interval  $[0, T]$  with fixed equal length, with  $\delta_i = i\delta$ ,  $i = 1, \dots, N$ ; and  $N = \frac{T}{\delta}$ . We also denote  $n$  by the number of generalised particles in the system. The approximating algorithm is then introduced as follows.

**Initialisation:** At time  $t = 0$ , the particle system consists of  $n$  Gaussian measures all with equal weights  $1/n$ , initial means  $v_j^n(0)$ , and initial variances  $\omega_j^n(0)$ , for  $j = 1, \dots, n$ ; denoted by  $\Gamma_{v_j^n(0), \omega_j^n(0)}$ . The approximation of  $\pi_0^{n,\varepsilon}$  has the form

$$\pi_0^{n,\varepsilon} \triangleq \frac{1}{n} \sum_{j=1}^n \Gamma_{v_j^n(0), \omega_j^n(0)}. \quad (3.1)$$

We will, for  $1 \leq j \leq n$ , choose the initial variances  $\omega_j^n(0) = \alpha\beta$  and be given the initial means  $v_j^n(0)$ , where  $\varepsilon$ ,  $\alpha$  and  $\beta$  are some parameters defined later in this section.

**Recursion:** During the interval  $t \in [i\delta, (i+1)\delta)$ ,  $i = 1, \dots, N$ , the approximation  $\pi^{n,\varepsilon}$  of the normalised conditional distribution  $\pi$  will take the form

$$\pi_t^{n,\varepsilon} \triangleq \sum_{j=1}^n \bar{a}_j^n(t) \Gamma_{v_j^n(t), \omega_j^n(t)}, \quad (3.2)$$

where  $v_j^n(t)$  denotes the mean and  $\omega_j^n(t)$  denotes the variance of the Gaussian measure  $\Gamma_{v_j^n(t), \omega_j^n(t)}$ , and  $\bar{a}_j^n(t)$  is the (unnormalised) weight of the particle, and

$$\bar{a}_j^n(t) = \frac{a_j^n(t)}{\sum_{k=1}^n a_k^n(t)}$$

is the normalised weight. Obviously, each particle is characterised by the triple process  $(a_j^n, v_j^n, \omega_j^n)$  which is chosen to evolve as

$$\begin{cases} a_j^n(t) = 1 + \int_{i\delta}^t a_j^n(s) h(v_j^n(s)) dY_s, \\ v_j^n(t) = v_j^n(i\delta) + \int_{i\delta}^t f(v_j^n(s)) ds + \sqrt{1-\alpha} \int_{i\delta}^t \sigma(v_j^n(s)) dV_s^{(j)}, \\ \omega_j^n(t) = \alpha \left( \beta + \int_{i\delta}^t \sigma^2(v_j^n(s)) ds \right), \end{cases} \quad (3.3)$$

where  $\{V^{(j)}\}_{j=1}^n$  are mutually independent Brownian motions and independent of  $Y$ . The parameter  $\alpha$  is a real number in the interval  $[0, 1]$ . Here we choose  $\alpha = n^{-\varepsilon}$ , where  $\varepsilon \in [0, \infty]$  is a non-negative parameter measuring the ‘‘Gaussianity’’ of the generalised particles. To be specific, the variance of each Gaussian (generalised) particle can be controlled by the value of  $\varepsilon$ . For  $\varepsilon = \infty$  ( $\alpha = 0$ ) we recover the classic particle approximation (see, for example, Chapter 9 in [1]) with the Gaussian measures degenerated to Dirac measures; for  $\varepsilon = 0$  ( $\alpha = 1$ ) we have the largest possible variances and the means of the Gaussian measures evolve deterministically (the stochastic term is eliminated). Therefore we can normally restrict ourselves to the cases where  $\varepsilon \in (0, \infty)$ . One of the purposes of this paper is to find the optimal value for  $\varepsilon$ . The parameter  $\beta$  is a positive real number, which we call the *smoothing parameter*, ensures that the approximating measure has smooth density at the branching/correction times.

**Correction:** At the end of the interval  $[i\delta, (i+1)\delta)$ , immediately prior to the correction step, each Gaussian measure is replaced by a random number of offsprings, which are Gaussian measures with mean  $X_j^n((i+1)\delta)$  and variance  $\alpha\beta$ , where the mean  $X_j^n$  is a normally distributed random variable, i.e.

$$X_j^n((i+1)\delta) \sim \mathcal{N}(v_j^n(i+1)\delta_-, \omega_j^n(i+1)\delta_-), \quad j = 1, \dots, n;$$

where by  $(i+1)\delta_-$  we denote the time immediately prior to correction. We denote by  $o_j^{n,(i+1)\delta}$  the number of ‘‘offsprings’’ produced by the  $j$ th generalised particle. The total number of offsprings is fixed to be  $n$  at each correcting event.

After correction all the particles are re-indexed from 1 to  $n$  and all of the unnormalised weights are re-initialised back to 1; and the particles evolve following (3.3) again. The recursion is repeated  $N$  times until we reach the terminal time  $T$ , where we obtain the approximation  $\pi_T^n$  of  $\pi_T$ .

We refer to [5] for a brief explanation why we should introduce correction mechanism. In the following we adopt the correction algorithm called the Multinomial Resampling to determine the number of offsprings  $\{o_j^n\}_{j=1}^n$  (see, for example, [6]). The multinomial resampling algorithm essentially consists of sampling  $n$  times with replacement at correction times. At branching times, we sample  $n$  times (with replacement) from the population of Gaussian random variables  $X_j^n((i+1)\delta)$  (with means  $v_j^n((i+1)\delta_-)$  and variances  $\omega_j^n((i+1)\delta_-)$ ,  $j = 1, \dots, n$  according to the multinomial probability distribution given by the corresponding normalised weights  $\bar{a}_j^n((i+1)\delta_-)$ ,  $j = 1, \dots, n$ . Therefore, by definition of multinomial distribution,  $o_j^{n,(i+1)\delta}$  is the number of times  $X_j^n((i+1)\delta)$  is chosen at time  $(i+1)\delta$ ; that is to say,  $o_j^{n,(i+1)\delta}$  is the number of offspring produced by this Gaussian random variable.

We then define the process  $\xi^n = \{\xi_t^n; t \geq 0\}$  by

$$\xi_t^n \triangleq \left( \prod_{i=1}^{\lfloor t/\delta \rfloor} \frac{1}{n} \sum_{j=1}^n a_j^{n,i\delta} \right) \left( \frac{1}{n} \sum_{j=1}^n a_j^n(t) \right).$$

Then  $\xi^n$  is a martingale and by Exercise 9.10 in [1] we know for any  $t \geq 0$  and  $p \geq 1$ , there exist two constants  $c_1^{t,p}$  and  $c_2^{t,p}$  which depend only on  $t, p$ , and  $\max_{k=1,\dots,m} \|h_k\|_{0,\infty}$ , such that

$$\sup_{n \geq 0} \sup_{s \in [0,t]} \tilde{\mathbb{E}}[(\xi_s^n)^p] \leq c_1^{t,p}, \quad (3.4)$$

and

$$\max_{j=1,\dots,n} \sup_{n \geq 0} \sup_{s \in [0,t]} \tilde{\mathbb{E}}[(\xi_s^n a_j^n(s))^p] \leq c_2^{t,p}. \quad (3.5)$$

We use the martingale  $\xi^n$  to linearise  $\pi^{n,\varepsilon}$ , to be specific, we define the measure-valued process  $\rho^{n,\varepsilon} = \{\rho_t^{n,\varepsilon} : t \geq 0\}$  to be

$$\rho_t^{n,\varepsilon} \triangleq \xi_t^n \pi_t^{n,\varepsilon} = \frac{\xi_{[t/\delta]\delta}^n}{n} \sum_{j=1}^n a_j^n(t) \Gamma_{v_j^n(t), \omega_j^n(t)}. \quad (3.6)$$

Define  $U = \{U_t^{n,\varepsilon} : t \geq 0\}$  to be the measure-valued process

$$U_t^{n,\varepsilon} \triangleq n^\varepsilon (\rho_t^{n,\varepsilon} - \rho_t), \quad (3.7)$$

and we aim to find an appropriate range for  $\varepsilon$  and show that, with the right choice of  $\varepsilon$ , the corresponding  $\{U^{n,\varepsilon}\}_n$  converges in distribution to a process  $U^\varepsilon$ , which is uniquely identified as the solution of a certain martingale problem. This implies that for any continuous and bounded test function,

$$\lim_{n \rightarrow \infty} n^\varepsilon (\rho_t^{n,\varepsilon}(\varphi) - \rho_t(\varphi)) = U_t^\varepsilon(\varphi); \quad (3.8)$$

hence the error of the approximations  $\rho_t^{n,\varepsilon}(\varphi)$  of  $\rho_t(\varphi)$  is roughly  $U_t^\varepsilon(\varphi)n^{-\varepsilon}$ .

By Proposition 4.1 in [5], we have

$$U_t^{n,\varepsilon}(\varphi) = U_0^{n,\varepsilon}(\varphi) + \int_0^t U_s^{n,\varepsilon}(A\varphi)ds + \int_0^t U_s^{n,\varepsilon}(h\varphi)dY_s + n^\varepsilon M_{[t/\delta]}^{n,\varphi} + n^\varepsilon B_t^{n,\varphi}, \quad (3.9)$$

in (3.9),

$$n^\varepsilon M_{[t/\delta]}^{n,\varphi} = n^{\varepsilon-1} \sum_{i=0}^{[t/\delta]} \xi_{i\delta}^n \sum_{j=1}^n \left[ o_j^{n,i\delta} \int_{\mathbb{R}} \varphi(x) \frac{e^{-\frac{(x-X_j^n(i\delta))^2}{2\alpha\beta}}}{\sqrt{2\pi\alpha\beta}} dx - n\bar{a}_j^n(i\delta-) \int_{\mathbb{R}} \varphi(x) \frac{e^{-\frac{(x-v_j^n(i\delta-))^2}{2\omega_j^n(i\delta-)}}}{\sqrt{2\pi\omega_j^n(i\delta-)}} dx \right], \quad (3.10)$$

$$n^\varepsilon B_t^{n,\varphi} = n^{\varepsilon-1} \sum_{j=1}^n \int_0^t \xi_{[s/\delta]\delta}^n a_j^n(s) \left[ R_{s,j}^1(\varphi)ds + R_{s,j}^2(\varphi)dY_s + R_{s,j}^3(\varphi)dV_s^{(j)} \right]; \quad (3.11)$$

where

$$R_{s,j}^1(\varphi) = \omega_j^n(s) \left[ \frac{1}{2}(f\varphi''')(v_j^n(s)) + \frac{\alpha}{4}(\sigma\varphi^{(4)})(v_j^n(s)) + 2\alpha\sigma^2(v_j^n(s))I_{4,j}^{(4)}(\varphi) - I_j(A\varphi) \right] \\ + (\omega_j^n(s))^2 \left[ f(v_j^n(s))I_{4,j}^{(5)}(\varphi) + \frac{\alpha\sigma^2(v_j^n(s))}{2\sqrt{\omega_j^n(s)}}I_{5,j}(\varphi) + \frac{1-\alpha}{2}\sigma^2(v_j^n(s))I_{4,j}^{(6)}(\varphi) \right], \quad (3.12)$$

$$R_{s,j}^2(\varphi) = \omega_j^n(s) \left[ \frac{1}{2}h(v_j^n(s))\varphi''(v_j^n(s)) - I_j(h\varphi) \right] + (\omega_j^n(s))^2 h(v_j^n(s))I_{4,j}^{(4)}(\varphi), \quad (3.13)$$

$$R_{s,j}^3(\varphi) = \sqrt{1-\alpha} \left[ \sigma(v_j^n(s))\varphi'(v_j^n(s)) + \frac{1}{2}\omega_j^n(s)\sigma(v_j^n(s))\varphi'''(v_j^n(s)) \right. \\ \left. + (\omega_j^n(s))^2\sigma(v_j^n(s))I_{4,j}^{(5)}(\varphi) \right]; \quad (3.14)$$



and

$$\begin{aligned}
I_{4,j}^{(k)}(\varphi) &= \int_{\mathbb{R}} \frac{y^4 e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \int_0^1 \varphi^{(k)} \left( v_j^n(s) + uy\sqrt{\omega_j^n(s)} \right) \frac{(1-u)^3}{6} du dy, \quad \text{for } k = 4, 5, 6; \\
I_{5,j}(\varphi) &= \int_{\mathbb{R}} \frac{y^5 e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \int_0^1 \varphi^{(5)} \left( v_j^n(s) + uy\sqrt{\omega_j^n(s)} \right) \frac{u(1-u)^3}{6} du dy; \\
I_j(\psi) &= \int_{\mathbb{R}} \frac{y^2 e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \int_0^1 (\psi)'' \left( v_j^n(s) + uy\sqrt{\omega_j^n(s)} \right) (1-u) du dy, \quad \text{for } \psi = A\varphi, h\varphi.
\end{aligned}$$

The machinery used to prove the convergence in distribution for  $U^{n,\varepsilon}$  consists of two steps. In step one we show the tightness property of  $U^{n,\varepsilon}$ . In step two we show that any convergent subsequence of  $U^{n,\varepsilon}$  has a limit  $U^\varepsilon$  (in distribution) that is the unique solution of a certain martingale problem. These two steps are done in the following two sections.

### Discussion on the parameter $\varepsilon$

Before proceeding to the proof of convergence in distribution, here we discuss the influence of  $\varepsilon$  on the convergence of the approximating algorithm. From Section 4 in [5] it can be concluded that the  $L^2$ -convergence rate of the Gaussian mixture approximation is  $(\frac{1}{n})^{\min\{2\varepsilon, 1\}}$ . It means that for  $\varepsilon \in (0, 1/2]$  the convergence rate becomes better as  $\varepsilon$  increases, and it then stays at  $n^{-1}$  for any  $\varepsilon > 1/2$ .

Following the proof of Lemma 4.7 in [5], it can be shown that, when  $\varepsilon > 1/2$ ,  $n^\varepsilon M_{[t/\delta]}^{n,\varphi}$  in (3.10) will diverge as  $n \rightarrow \infty$ . Therefore the limit (in distribution) of the measure valued process  $\{U^{n,\varepsilon}\}_n$  does not exist when  $\varepsilon > 1/2$ , and the central limit theorem for the Gaussian mixture approximation can only be possibly obtained when  $\varepsilon \in (0, 1/2]$ .

As we will see in the following two sections, the essence of the analysis and proofs of the convergence in distribution is the same for different  $\varepsilon$ , except for some notational changes. In other words, the central limit theorem can be proven for all  $\varepsilon \in (0, 1/2]$  in the same manner, and the choice of  $\varepsilon$  will not have a crucial influence on the proof. We therefore choose  $\varepsilon = 1/2$  in the remaining of the paper, since it gives us the optimal  $L^2$ -convergence rate  $(1/n)$  of the approximating algorithm. Thus, with no risk of abuse of notations, we can eliminate the superscript  $\varepsilon$  for  $U^{n,\varepsilon}$ ,  $\pi^{n,\varepsilon}$  and  $\rho^{n,\varepsilon}$ , and simply write them as  $U^n$ ,  $\pi^n$  and  $\rho^n$  from next section to ease notations.

## 4 Step One: Tightness

In this section we prove the tightness of the measure-valued process  $\{U_t^n; t \geq 0\}$ . It is possible to obtain the tightness and convergence in distribution results by endowing  $\mathcal{M}_F(\mathbb{R})$  with the weak topology. In this topology a sequence of finite measures  $\{\mu^n\}_{n \in \mathbb{N}} \subset \mathcal{M}_F(\mathbb{R})$  converges to  $\mu \in \mathcal{M}_F(\mathbb{R})$  if and only if for a set  $\mathcal{S}(\varphi)$  of test functions,  $\mu^n(\varphi)$  converges to  $\mu(\varphi)$  for all  $\varphi \in \mathcal{S}(\varphi)$ .  $\mathcal{S}(\varphi)$  can be taken to be  $C_b^m(\mathbb{R})$  for any  $m \geq 1$ .

Before proceeding further discussion on  $U^n$ , we define the metric on  $\mathcal{M}_F(\mathbb{R})$  which generates the weak topology. Let  $\varphi_0 = 1$  and  $\{\varphi_i\}_{i \geq 0}$  be a sequence of functions which are dense in the space of continuous functions with compact support on  $\mathbb{R}$ . Then the metric  $d_{\mathcal{M}}$  is defined as

$$d_{\mathcal{M}} : \mathcal{M}_F(\mathbb{R}) \times \mathcal{M}_F(\mathbb{R}) \rightarrow [0, \infty), \quad d_{\mathcal{M}}(\mu, \nu) = \sum_{i=0}^{\infty} \frac{|\mu(\varphi_i) - \nu(\varphi_i)|}{2^i \|\varphi_i\|_{0,\infty}};$$



and  $d_{\mathcal{M}}$  generates the weak topology on  $\mathcal{M}_F(\mathbb{R})$  in the sense that  $\mu^n$  converges weakly to  $\mu$  if and only if  $\lim_{n \rightarrow \infty} d_{\mathcal{M}}(\mu^n, \mu) = 0$  as  $\{\varphi_i\}_{i \geq 0}$  is a convergence determining set of functions over  $\mathcal{M}_F(\mathbb{R})$ .

However, the space  $(D_{\mathcal{M}_F(\mathbb{R})}[0, \infty), d_{\mathcal{M}})$  is separable but not complete under this metric because its underlying space  $(\mathcal{M}_F(\mathbb{R}), d_{\mathcal{M}})$  is separable but not complete. This inconvenience makes us unable to make use of Prohorov's Theorem (see, for example, Theorem 2.4.7 in [15]). In order to tackle this problem, we consider the one-point compactification of  $\mathbb{R}$

$$\overline{\mathbb{R}} \triangleq \mathbb{R} \cup \{\infty\},$$

Then we embed the space  $D_{\mathcal{M}_F(\mathbb{R})}[0, \infty)$  into the complete and separable space  $D_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, \infty)$  by defining a map such that

$$\mu \in \mathcal{M}_F(\mathbb{R}) \rightarrow \overline{\mu} \in \mathcal{M}_F(\overline{\mathbb{R}}) \quad \text{and} \quad \overline{\mu}(A) = \mu(A \cap \mathbb{R}), \quad \forall A \in \overline{\mathbb{R}}.$$

The family  $\{U_t^n\}$  can then be viewed as a stochastic process with sample paths in the complete and separable space  $D_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, \infty)$ , or as a random variable with values in the space  $\mathcal{P}(D_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, \infty))$  – the space of probability measures over  $D_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, \infty)$ .

We are now ready to show that the family of processes  $\{U^n\}$  is tight on  $[0, T]$  for all  $T > 0$ . In other words, let  $\{\tilde{\mathbb{P}}_n\} \subset \mathcal{P}(D_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, T])$  be the family of associated probability distributions of  $U^n$ ; in other words,  $\tilde{\mathbb{P}}_n(B) = \tilde{\mathbb{P}}_n(U^n \in B)$  for all  $B \in \mathcal{B}(D_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, T])$ . We aim to show that  $\{\tilde{\mathbb{P}}_n\}$  is relatively compact and hence, by Prohorov's Theorem, tight. To be specific, we will make use of the following theorem (Theorem 2.1 in [22]):

**Theorem 4.1.** *A family of probabilities  $\{\tilde{\mathbb{P}}_n\}_n \subset \mathcal{P}(D_{\mathcal{M}_F(\overline{\mathbb{R}^d})}[0, T])$  is tight, if there exists a dense sequence  $\{\tilde{f}_k\}_{k \geq 0}$  in  $C_b(\overline{\mathbb{R}^d})$  such that for each  $k \in \mathbb{N}$ ,  $\{\pi_{\tilde{f}_k} \tilde{\mathbb{P}}_n\}_n \subset \mathcal{P}(D_{\overline{\mathbb{R}}}^d[0, T])$  is a tight sequence of probabilities; where  $\pi_{\tilde{f}_k} : \mathcal{M}_F(\overline{\mathbb{R}^d}) \rightarrow \overline{\mathbb{R}}$  is defined by  $\pi_{\tilde{f}_k}(\mu) = \mu(\tilde{f}_k)$  for  $\mu \in \mathcal{M}_F(\overline{\mathbb{R}^d})$ .*

In the remaining of this section, because of the definition of the distance  $d_{\mathcal{M}}$ , we choose  $(\tilde{f}_k)_{k \geq 0}$  to be defined as follows:  $\tilde{f}_0 \equiv 1$ , and  $\tilde{f}_k$  ( $k \geq 1$ ) is chosen so that  $\tilde{f}_k|_{\mathbb{R}}$  is a dense sequence in  $\mathcal{C}_b^6(\mathbb{R})$ , the space of six times differentiable continuous functions on  $\mathbb{R}$ , vanishing at infinity with continuous partial derivatives up to and including the sixth order.

According to Theorem 4.1, it suffices to prove the tightness result for  $\{\pi_{\tilde{f}_k} \tilde{\mathbb{P}}_n\}_n$ . We will make use of the following criteria, which can be found in [13], to show that  $\{\pi_{\tilde{f}_k} U^n\}_n = \{U^n(\tilde{f}_k)\}_n$  is tight, and then the tightness of  $\{\pi_{\tilde{f}_k} \tilde{\mathbb{P}}_n\}$  follows by applying Theorem 4.1.

**Theorem 4.2** (Kurtz's criteria of relative compactness). *Let  $(E, d)$  be a separable and complete metric space and let  $\{X^n\}_{n \in \mathbb{N}}$  be a sequence of processes with sample paths in  $D_E[0, \infty)$ . Suppose that for every  $\eta > 0$  and rational  $t$ , there exists a compact set  $\Gamma_{\eta, t}$  such that*

$$\sup_n \mathbb{P}(X_t^n \notin \Gamma_{\eta, t}) \leq \eta. \quad (4.1)$$

*Then  $\{X^n\}_{n \in \mathbb{N}}$  is relatively compact if and only if the following conditions hold:*

- For each  $T' > 0$ , there exists  $\zeta > 0$  and a family  $\{\gamma^n(\Delta) : 0 < \Delta < 1\}$  of non-negative random variables

$$\tilde{\mathbb{E}} \left[ (1 \wedge d(X_{t+u}^n, X_t^n))^{\zeta} (1 \wedge d(X_t^n, X_{t-v}^n))^{\zeta} | \mathcal{F}_t \right] \leq \tilde{\mathbb{E}} [\gamma^n(\Delta) | \mathcal{F}_t] \quad (4.2)$$

*for  $0 \leq t \leq T'$ ,  $0 \leq u \leq \Delta$  and  $0 \leq v \leq \Delta \wedge t$ ;*

- For  $\gamma^n(\Delta)$ , we have

$$\lim_{\Delta \rightarrow 0} \limsup_{n \rightarrow \infty} \tilde{\mathbb{E}}[\gamma^n(\Delta)] = 0; \quad (4.3)$$

- At the initial time

$$\lim_{\Delta \rightarrow 0} \limsup_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[ (1 \wedge d(X_\Delta^n, X_0^n))^\zeta \right] = 0. \quad (4.4)$$

To justify (4.1), we need to prove the following lemma:

**Lemma 4.3.** *For all  $\eta > 0$ , there exists a constant  $\bar{\beta}$  such that for the associated probabilities  $\{\pi_{\tilde{f}_k} \tilde{\mathbb{P}}_n\}$  of  $\{\pi_{\tilde{f}_k} U^n\}$  and  $A = \{x \in D_{\mathbb{R}}[0, T] : \sup_{t \in [0, T]} |x(t)| > \bar{\beta}\}$ , we have*

$$\pi_{\tilde{f}_k} \tilde{\mathbb{P}}_n(A) \leq \eta. \quad (4.5)$$

*Proof.* Note that  $\pi_{\tilde{f}_k} U_t^n = U_t^n(\tilde{f}_k)$ , so that

$$\begin{aligned} \pi_{\tilde{f}_k} \tilde{\mathbb{P}}_n(A) &= \tilde{\mathbb{P}}_n \pi_{\tilde{f}_k}^{-1}(A) = \tilde{\mathbb{P}}_n \left( U^n \in D_{\mathcal{M}_F}[0, T] : \sup_t |U_t^n(\tilde{f}_k)| > \bar{\beta} \right) \\ &= \tilde{\mathbb{P}}_n \left( U^n \in D_{\mathcal{M}_F}[0, T] : \sup_t |\sqrt{n}(\rho_t^n(\tilde{f}_k) - \rho_t(\tilde{f}_k))| > \bar{\beta} \right) \leq \frac{\Lambda_T^n(\tilde{f}_k)}{\bar{\beta}^2}, \end{aligned} \quad (4.6)$$

where  $\Lambda_T^n(\tilde{f}_k) = \tilde{\mathbb{E}} \left[ \sup_t \left( \sqrt{n}(\rho_t^n(\tilde{f}_k) - \rho_t(\tilde{f}_k)) \right)^2 \right]$ .

It suffices to show that  $\Lambda_T^n(\tilde{f}_k)$  is bounded above by a constant independent of  $n$ , which is an immediate consequence of Jensen's inequality and Theorem 4.18 in [5]. Then we choose

$$\bar{\beta}^2 = \frac{\eta}{\Lambda_T^n(\tilde{f}_k)}$$

and the proof is complete.  $\square$

In order to prove the tightness of  $\{U^n(\tilde{f}_k)\}_n$ , we need to show that  $\{U^n(\tilde{f}_k)\}_n$  satisfies (4.2), (4.3) and (4.4). We prove these by showing that each of the increments of the process appearing on the right hand side of (3.9) satisfies similar bounds.

In the following we will choose  $\Delta$  to be sufficiently small. To be specific, we let  $\Delta < \frac{\delta}{2}$ , where  $\delta$  is the time length between two resampling events. This ensures that either  $[t - \Delta, t]$  or  $[t, t + \Delta]$  does not contain a resampling event, in other words, there is at most one resampling event in  $[t, t + u]$  and  $[t - v, t]$ , where  $0 \leq u \leq \Delta$  and  $0 \leq v \leq \Delta \wedge t$ .

If the resampling happens only in the interval  $[t - v, t]$ , and obtain

$$\tilde{\mathbb{E}} \left[ (1 \wedge d(X_{t+u}^n, X_t^n))^\zeta (1 \wedge d(X_t^n, X_{t-v}^n))^\zeta | \mathcal{F}_t \right] \leq \tilde{\mathbb{E}} \left[ (1 \wedge d(X_{t+u}^n, X_t^n))^\zeta | \mathcal{F}_t \right].$$

Therefore in order to determine  $\gamma^n(\Delta)$  and shows that (4.2) is satisfied by  $\{U^n(\tilde{f}_k)\}_n$ , it suffices to find an appropriate  $\gamma^n(\Delta)$  for  $\zeta = 2$  and show that

$$\tilde{\mathbb{E}} \left[ \left( 1 \wedge d(U_{t+u}^n(\tilde{f}_k), U_t^n(\tilde{f}_k)) \right)^2 | \mathcal{F}_t \right] \leq \tilde{\mathbb{E}} [\gamma^n(\Delta) | \mathcal{F}_t]. \quad (4.7)$$

This will be done in the following proposition.

**Proposition 4.4.** *Let  $k \in \mathbb{N}$ , and we further assume that  $\tilde{f}_k \in C_b^6(\mathbb{R})$ , and Assumption (A) holds. Let the length between two resampling events  $\delta$  be fixed and let  $\alpha \propto \frac{1}{\sqrt{n}}$ . Define the family  $\{\gamma_u^n(\Delta) : 0 < \Delta < 1\}$  of non-negative random variables*

$$\begin{aligned} \gamma^n(\Delta) \triangleq & 3n\Delta^2 \sup_{s \in [t, t+u]} \left( \rho_s^n(A\tilde{f}_k) - \rho_s(A\tilde{f}_k) \right)^2 + 3n\Delta \sup_{s \in [t, t+u]} \left( \rho_s^n(h\tilde{f}_k) - \rho_s(h\tilde{f}_k) \right)^2 \\ & + \frac{3\Delta}{n} C_\gamma \|\tilde{f}_k\|_{6,\infty}^2 \sum_{j=1}^n \sup_{s \in [t, t+u]} \left( \xi_{i\delta}^n a_j^n(s) \right)^2, \end{aligned} \quad (4.8)$$

where  $C_\gamma$  is a constant independent of  $n$ . By Theorem 4.18 in [5], we know that

$$\sup_{s \in [t, t+u]} n \left( \rho_s^n(A\tilde{f}_k) - \rho_s(A\tilde{f}_k) \right)^2 \quad \text{and} \quad \sup_{s \in [t, t+u]} n \left( \rho_s^n(h\tilde{f}_k) - \rho_s(h\tilde{f}_k) \right)^2$$

are bounded and independent of  $\Delta$ . Then we have

$$\tilde{\mathbb{E}} \left[ 1 \wedge d(U_{t+u}^n(\tilde{f}_k), U_t^n(\tilde{f}_k))^2 | \mathcal{F}_t \right] \leq \tilde{\mathbb{E}} [\gamma^n(\Delta) | \mathcal{F}_t]. \quad (4.9)$$

*Proof.* Bearing in mind that there is no resampling event within  $[t, t+u]$ , thus  $[(t+u)/\delta] = [t/\delta]$  and

$$M_{[(t+u)/\delta]}^{n, \tilde{f}_k} - M_{[t/\delta]}^{n, \tilde{f}_k} = 0.$$

Therefore we have that

$$\begin{aligned} & \tilde{\mathbb{E}} \left[ 1 \wedge d(U_{t+u}^n(\tilde{f}_k), U_t^n(\tilde{f}_k))^2 | \mathcal{F}_t \right] \leq \tilde{\mathbb{E}} \left[ |U_{t+u}^n(\tilde{f}_k) - U_t^n(\tilde{f}_k)|^2 | \mathcal{F}_t \right] \\ & = \tilde{\mathbb{E}} \left[ \left| \sqrt{n} \left( \left( \rho_{t+u}^n(\tilde{f}_k) - \rho_{t+u}(\tilde{f}_k) \right) - \left( \rho_t^n(\tilde{f}_k) - \rho_t(\tilde{f}_k) \right) \right) \right|^2 | \mathcal{F}_t \right] \\ & \leq 3n \left\{ \tilde{\mathbb{E}} \left[ \left( \int_t^{t+u} (\rho_s^n(A\tilde{f}_k) - \rho_s(A\tilde{f}_k)) ds \right)^2 | \mathcal{F}_t \right] + \tilde{\mathbb{E}} \left[ \left( \int_t^{t+u} (\rho_s^n(h\tilde{f}_k) - \rho_s(h\tilde{f}_k)) dY_s \right)^2 | \mathcal{F}_t \right] \right. \\ & \quad \left. + \frac{1}{n^2} \tilde{\mathbb{E}} \left[ \left( \sum_{j=1}^n \int_t^{t+u} \xi_{i\delta}^n a_j^n(s) \left[ R_{s,j}^1(\tilde{f}_k) ds + R_{s,j}^2(\tilde{f}_k) dY_s + R_{s,j}^3(\tilde{f}_k) dV_s^{(j)} \right] \right)^2 | \mathcal{F}_t \right] \right\}. \end{aligned} \quad (4.10)$$

We examine each of the terms in (4.10) and observe the following:

For the first term in (4.10), by Jensen's inequality, we have

$$\tilde{\mathbb{E}} \left[ \left( \sqrt{n} \int_t^{t+u} (\rho_s^n(A\tilde{f}_k) - \rho_s(A\tilde{f}_k)) ds \right)^2 | \mathcal{F}_t \right] \leq nu^2 \sup_{s \in [t, t+u]} \tilde{\mathbb{E}} \left[ \left( \rho_s^n(A\tilde{f}_k) - \rho_s(A\tilde{f}_k) \right)^2 | \mathcal{F}_t \right]. \quad (4.11)$$

For the second term in (4.10),

$$\tilde{\mathbb{E}} \left[ \left( \int_t^{t+u} \sqrt{n} (\rho_s^n(h\tilde{f}_k) - \rho_s(h\tilde{f}_k)) dY_s \right)^2 | \mathcal{F}_t \right] \leq un \sup_{s \in [t, t+u]} \tilde{\mathbb{E}} \left[ \left( \rho_s^n(h\tilde{f}_k) - \rho_s(h\tilde{f}_k) \right)^2 | \mathcal{F}_t \right]. \quad (4.12)$$

For the remaining terms in (4.10), note that

$$R_{s,j}^1(\tilde{f}_k) \leq C_1 \alpha \delta \|\tilde{f}_k\|_{6,\infty} \leq \frac{C_1}{n} \|\tilde{f}_k\|_{6,\infty},$$

we then have

$$n \frac{1}{n^2} \tilde{\mathbb{E}} \left[ \left( \sum_{j=1}^n \int_t^{t+u} \xi_{i\delta}^n a_j^n(s) \left[ R_{s,j}^1(\tilde{f}_k) ds \right] \right)^2 \middle| \mathcal{F}_t \right] \leq u \frac{C_1^2}{n} \|\tilde{f}_k\|_{6,\infty}^2 \sum_{j=1}^n \sup_{s \in [t, t+u]} \tilde{\mathbb{E}} \left[ \left( \xi_{i\delta}^n a_j^n(s) \right)^2 \middle| \mathcal{F}_t \right]; \quad (4.13)$$

and also note that

$$R_{s,j}^2(\tilde{f}_k) \leq C_2 \alpha \delta \|\tilde{f}_k\|_{4,\infty} \leq \frac{C_2}{n} \|\tilde{f}_k\|_{4,\infty},$$

then we have

$$n \frac{1}{n^2} \tilde{\mathbb{E}} \left[ \left( \sum_{j=1}^n \int_t^{t+u} \xi_{i\delta}^n a_j^n(s) R_{s,j}^2(\varphi) dY_s \right)^2 \middle| \mathcal{F}_t \right] \leq u \frac{C_2^2}{n} \|\tilde{f}_k\|_{4,\infty}^2 \sum_{j=1}^n \sup_{s \in [t, t+u]} \tilde{\mathbb{E}} \left[ \left( \xi_{i\delta}^n a_j^n(s) \right)^2 \middle| \mathcal{F}_t \right]; \quad (4.14)$$

and finally since

$$R_{s,j}^3(\tilde{f}_k) \leq (C_0 + C_3 \alpha \delta) \|\tilde{f}_k\|_{5,\infty} \leq (C_0 + C_3) \|\tilde{f}_k\|_{5,\infty},$$

we have that

$$n \frac{1}{n^2} \tilde{\mathbb{E}} \left[ \left( \sum_{j=1}^n \int_t^{t+u} \xi_{i\delta}^n a_j^n(s) R_{s,j}^3(\varphi) dV_s^{(j)} \right)^2 \middle| \mathcal{F}_t \right] \leq \frac{u}{n} (C_0 + C_3)^2 \|\tilde{f}_k\|_{5,\infty}^2 \sum_{j=1}^n \sup_{s \in [t, t+u]} \tilde{\mathbb{E}} \left[ \left( \xi_{i\delta}^n a_j^n(s) \right)^2 \middle| \mathcal{F}_t \right]. \quad (4.15)$$

Therefore, considering the bounds in the right hand sides of (4.11), (4.12), (4.13), (4.14), and (4.15); we can define  $\gamma^n(\Delta)$  as in (4.8) by letting

$$C_\gamma = C_1^2 + C_2^2 + (C_0 + C_3)^2.$$

By virtue of (4.10), we know that (4.9) is satisfied.  $\square$

The above discussion defines  $\gamma^n(\Delta)$  and shows that (4.2) is satisfied for  $\{U^n(\tilde{f}_k)\}_n$ . The following proposition shows that  $\gamma^n(\Delta)$  defined in (4.8) satisfies (4.3).

**Proposition 4.5.**  *$\gamma^n(\Delta)$  defined in (4.8) has the following property*

$$\lim_{\Delta \rightarrow 0} \limsup_{n \rightarrow \infty} \tilde{\mathbb{E}} [\gamma^n(\Delta)] = 0. \quad (4.16)$$

*Proof.* We show this by looking at the expectation of each term in (4.8).

For the first term, by Theorem 4.18 in [5]

$$\tilde{\mathbb{E}} \left[ n \Delta^2 \sup_{s \in [t, t+u]} \left( \rho_s^n(A\tilde{f}_k) - \rho_s(A\tilde{f}_k) \right)^2 \right] \leq \Delta^2 c^T \left\| A\tilde{f}_k \right\|_{m+2,\infty}^2 \rightarrow 0, \quad \text{as } \Delta \rightarrow 0.$$

Similarly, for the second term,

$$\tilde{\mathbb{E}} \left[ n\Delta \sup_{s \in [t, t+u]} \left( \rho_s^n(h\tilde{f}_k) - \rho_s(h\tilde{f}_k) \right)^2 \right] \leq \Delta \tilde{c}^T \left\| h\tilde{f}_k \right\|_{m+2, \infty}^2 \rightarrow 0, \quad \text{as } \Delta \rightarrow 0.$$

For the remaining term, again note that  $(\alpha\delta)^2 \sim 1/n$ , and

$$\tilde{\mathbb{E}} \left[ \sum_{j=1}^n \sup_{s \in [t, t+u]} \left( \xi_{i\delta}^n a_j^n(s) \right)^2 \right] = \sum_{j=1}^n \tilde{\mathbb{E}} \left[ \sup_{s \in [t, t+u]} \left( \xi_{i\delta}^n a_j^n(s) \right)^2 \right] \leq n c_2^{t,2}.$$

Thus

$$\frac{\Delta}{n} C_{\gamma^n} \|\tilde{f}_k\|_{6, \infty}^2 \sum_{j=1}^n \tilde{\mathbb{E}} \left[ \sup_{s \in [t, t+u]} \left( \xi_{i\delta}^n a_j^n(s) \right)^2 \right] \leq \frac{\Delta}{n} \|\tilde{f}_k\|_{6, \infty}^2 n c_2^{t,2} = \Delta c_2^{t,2} \|\tilde{f}_k\|_{6, \infty}^2 \rightarrow 0, \quad \text{as } \Delta \rightarrow 0.$$

This completes the proof.  $\square$

The following proposition shows that (4.4) holds for  $\{U^n(\tilde{f}_k)\}$ .

**Proposition 4.6.** *For each  $k \in \mathbb{N}$ , we have*

$$\lim_{\Delta \rightarrow 0} \limsup_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[ \left( 1 \wedge d(U_\Delta^n(\tilde{f}_k), U_0^n(\tilde{f}_k)) \right)^2 \right] = 0. \quad (4.17)$$

*Proof.* The result follows immediately by continuity of  $\{U^n(\tilde{f}_k)\}_n$  at the initial time 0.  $\square$

**Theorem 4.7.** *The measure-valued processes  $\{U_t^n : t \in [0, T]\}_{n \geq 1}$  forms a tight sequence.*

*Proof.* Lemma 4.3, Propositions 4.4 – 4.6 state that all the conditions in Theorem 4.2 are satisfied. Then by Theorem 4.2 we know that  $\{\pi_{\tilde{f}_k} U^n\}_n$  is tight, which implies that  $\{\pi_{\tilde{f}_k} \tilde{\mathbb{P}}_n\}_n$  forms a tight sequence on  $\mathcal{P}(D_{\mathbb{R}}[0, T])$ ; then by Theorem 4.1 we know  $\{\tilde{\mathbb{P}}_n\}$  forms a tight sequence on  $\mathcal{P}(D_{\mathcal{M}_F(\mathbb{R}^d)}[0, T])$ . By definition we can then conclude the following tightness result.  $\square$

**Remark 4.8.** *If we assume that the resampling happens only in  $[t, t+u]$ , then by exactly the same discussion as above (except that we replace  $s \in [t, t+u]$  by  $s \in [t-v, u]$ ), we can also obtain the tightness for the process  $\{U_t^n\}_{n \geq 1}$ .*

## 5 Step Two: Limits of Convergent Subsequences

In this section we show that  $\{U^n\}_n$  converges in distribution to a uniquely determined process  $U$ . The strategy of the proof of the convergence in distribution is as follows: Since the sequence of the measure-valued process  $\{U^n\}_n$  is tight, then any subsequence  $\{U^{n_k}\}_k$  of  $\{U^n\}_n$  contains a convergent sub-subsequence  $\{U^{n_{k_l}}\}_l$ . We will prove that any convergent subsequence has a weak limit  $U$  which is the unique solution of (5.4). This ensures that the entire sequence  $\{U^n\}_n$  is convergent and its weak limit is the solution  $U$  of (5.4).

We need the following preliminary result.

**Lemma 5.1.** Let  $\varphi \in C_b^m(\overline{\mathbb{R}})$  ( $m \geq 6$ ) be a test function, and define the measure-valued processes

$$\begin{aligned}\tilde{\rho}_t^{n,1} &\triangleq \frac{1}{n} \sum_{j=1}^n \xi_{i\delta}^n a_j^n(t) \delta_{v_j^n(t)} = \sum_{j=1}^n \xi_t^n \bar{a}_j^n(t) \delta_{v_j^n(t)}, \\ \tilde{\rho}_t^{n,2} &\triangleq \frac{1}{n} \sum_{j=1}^n \{\xi_{i\delta}^n a_j^n(t)\}^2 \delta_{v_j^n(t)} = n \sum_{j=1}^n \{\xi_t^n \bar{a}_j^n(t)\}^2 \delta_{v_j^n(t)}.\end{aligned}\quad (5.1)$$

then for any  $t \in [0, T]$ ,

$$\tilde{\rho}_t^{n,1} \rightarrow \tilde{\rho}_t^1, \quad \tilde{\rho}_t^{n,2} \rightarrow \tilde{\rho}_t^2, \quad \tilde{\mathbb{P}} - a.s.,$$

where  $\tilde{\rho}^1$  is the solution of the Zakai equation, and  $\tilde{\rho}^2$  is the measure-valued process satisfying, for any  $\varphi \in \mathcal{D}(A)$ ,

$$\tilde{\rho}_t^2(\varphi) = \pi_0(\varphi) + \int_0^t \left\{ \rho_s(\mathbf{1}) \rho_s(A\varphi) + \rho_s(h) \rho_s(h\varphi) \right\} ds + \int_0^t \left\{ \rho_s(\mathbf{1}) \rho_s(h\varphi) + \rho_s(h) \rho_s(\varphi) \right\} dY_s. \quad (5.2)$$

*Proof.* The proof is identical to that of  $\rho_t^n$  converging to  $\rho_t$ , which is included in [5].  $\square$

**Proposition 5.2.** For any  $\varphi \in C_b^6(\overline{\mathbb{R}})$ , let  $\Lambda^\varphi$  be the process defined by

$$\begin{aligned}\Lambda_t^\varphi &= \sum_{i=1}^{\lfloor t/\delta \rfloor} \rho_{i\delta}(\mathbf{1}) \sqrt{\pi_{i\delta}(\varphi^2) - (\pi_{i\delta}(\varphi))^2} \Upsilon_i + c_\omega \int_0^t \tilde{\rho}_s^1(\Psi\varphi) ds \\ &\quad + c_\omega \int_0^t (\tilde{\rho}_s^1(h\varphi'' - (h\varphi)')) dB_s^{(2)} + \int_0^t \sqrt{\tilde{\rho}_s^2((\sigma\varphi')^2)} dB_s^{(3)}\end{aligned}\quad (5.3)$$

for  $t \in [0, T]$ . In (5.3),  $\{\Upsilon_i\}_{i \in \mathbb{N}}$  is a sequence of independent identically distributed, standard normal random variables, and  $\left\{ \sqrt{\pi_{i\delta}(\varphi^2) - (\pi_{i\delta}(\varphi))^2} \Upsilon_i \right\}_i$  are mutually independent given the  $\sigma$ -algebra  $\mathcal{Y}$ .  $c_\omega$  is a constant independent of  $n$ , and the operator  $\Psi$  is defined by

$$\Psi\varphi = \frac{f\varphi'''}{2} + \frac{\sigma\varphi^{(4)}}{4} - \frac{3(A\varphi)''}{2}.$$

$B^{(2)}$  and  $B^{(3)}$  are two independent standard Brownian motion both independent of the observation  $Y$ .

If  $U$  is a  $\mathcal{D}_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, \infty)$ -valued process such that for  $\varphi \in C_b^6(\overline{\mathbb{R}})$

$$U_t(\varphi) = U_0(\varphi) + \int_0^t U_s(A\varphi) ds + \int_0^t U_s(h\varphi) dY_s + \Lambda_t^\varphi, \quad (5.4)$$

then  $U$  is pathwise unique. That is, for any two strong solutions  $U^1$  and  $U^2$  of (5.4) with common initial value  $\mathbb{P}[U_0^1 = U_0^2] = 1$ , the two processes are indistinguishable, i.e.  $\mathbb{P}[U_t^1 = U_t^2; t \in [0, T]] = 1$ .

*Proof.* The argument here is similar to Theorem 2.21 and Remark 3.4 in [20]. Firstly, it can be seen that the first, third and fourth terms of (5.3) are martingales while the second term is not a martingale.

Suppose there exist two solutions  $U^1$  and  $U^2$  of (5.4). Then take  $\varphi \in C_b^6(\mathbb{R})$ , we have

$$U_t^i(\varphi) = U_0^i(\varphi) + \int_0^t U_s^i(A\varphi)ds + \int_0^t U_s^i(h\varphi)dY_s + \Lambda_t^\varphi, \quad i = 1, 2. \quad (5.5)$$

For  $i, j = \{1, 2\}$  let  $U^{ij}(\varphi_1, \varphi_2) \triangleq \tilde{\mathbb{E}}[U^i(\varphi_1)U^j(\varphi_2)]$ , for  $\varphi_1, \varphi_2 \in C_b^6(\overline{\mathbb{R}})$ .

By Itô's formula we have

$$\begin{aligned} U^{12}(\varphi_1, \varphi_2) &= \int_0^t U^{12}(\varphi_1, A\varphi_2)ds + \int_0^t U^{12}(A\varphi_1, \varphi_2)ds + \int_0^t U^{12}(h\varphi_1, h\varphi_2)ds \\ &\quad + \int_0^t \tilde{\mathbb{E}} \left[ U_s^1(\varphi_1) \tilde{\rho}_s^1(\Psi\varphi_2) + U_s^2(\varphi_2) \tilde{\rho}_s^1(\Psi\varphi_1) \right] ds \\ &\quad + \int_0^t \tilde{\mathbb{E}} \left[ \sqrt{\tilde{\rho}_s^2((\sigma\varphi_1)^2) \tilde{\rho}_s^2((\sigma\varphi_2)^2)} + \tilde{\rho}_s^1(h\varphi_1'' - (h\varphi_1)'') \tilde{\rho}_s^1(h\varphi_2'' - (h\varphi_2)'') \right] ds \\ &\quad + \tilde{\mathbb{E}} \left[ \sum_{i=0}^{\lfloor t/\delta \rfloor} \tilde{\mathbb{E}} \left[ (\rho_{i\delta}(\mathbf{1}))^2 (\pi_{i\delta-}(\varphi_1\varphi_2) - \pi_{i\delta-}(\varphi_1)\pi_{i\delta-}(\varphi_2)) \mid \mathcal{F}_{i\delta-} \right] \right]; \end{aligned}$$

$$\begin{aligned} U^{11}(\varphi_1, \varphi_2) &= \int_0^t U^{11}(\varphi_1, A\varphi_2)ds + \int_0^t U^{11}(A\varphi_1, \varphi_2)ds + \int_0^t U^{11}(h\varphi_1, h\varphi_2)ds \\ &\quad + \int_0^t \tilde{\mathbb{E}} \left[ U_s^1(\varphi_1) \tilde{\rho}_s^1(\Psi\varphi_2) + U_s^1(\varphi_2) \tilde{\rho}_s^1(\Psi\varphi_1) \right] ds \\ &\quad + \int_0^t \tilde{\mathbb{E}} \left[ \sqrt{\tilde{\rho}_s^2((\sigma\varphi_1)^2) \tilde{\rho}_s^2((\sigma\varphi_2)^2)} + \tilde{\rho}_s^1(h\varphi_1'' - (h\varphi_1)'') \tilde{\rho}_s^1(h\varphi_2'' - (h\varphi_2)'') \right] ds \\ &\quad + \tilde{\mathbb{E}} \left[ \sum_{i=0}^{\lfloor t/\delta \rfloor} \tilde{\mathbb{E}} \left[ (\rho_{i\delta}(\mathbf{1}))^2 (\pi_{i\delta-}(\varphi_1\varphi_2) - \pi_{i\delta-}(\varphi_1)\pi_{i\delta-}(\varphi_2)) \mid \mathcal{F}_{i\delta-} \right] \right]; \end{aligned}$$

and similarly for  $U^{21}(\varphi_1, \varphi_2)$  and  $U^{22}(\varphi_1, \varphi_2)$ .

Let

$$v_t = (U_t^{12} - U_t^{11}) + (U_t^{21} - U_t^{22}), \quad (5.6)$$

it then follows that

$$v_t(\varphi_1, \varphi_2) = \int_0^t v_s(\varphi_1, A\varphi_2)ds + \int_0^t v_s(A\varphi_1, \varphi_2)ds + \int_0^t v_s(h\varphi_1, h\varphi_2)ds; \quad (5.7)$$

and  $v_0(\varphi_1, \varphi_2) = 0$ .

It follows by Theorem 2.21(i) and Remark 3.4 in [20] that (5.7) has a unique solution and since (5.7) is a homogeneous equation beginning at 0. Then we have  $v_t(\varphi_1, \varphi_2) \equiv 0$ , which implies

$$(U_t^{11} - U_t^{12}) + (U_t^{22} - U_t^{21}) = 0,$$

that is to say, for  $\varphi_1 = \varphi_2 = \varphi$

$$\tilde{\mathbb{E}} [U_t^1(\varphi)U_t^1(\varphi) - U_t^1(\varphi)U_t^2(\varphi)] + \tilde{\mathbb{E}} [U_t^2(\varphi)U_t^2(\varphi) - U_t^2(\varphi)U_t^1(\varphi)] = \tilde{\mathbb{E}} [(U_t^1(\varphi) - U_t^2(\varphi))^2] = 0;$$

and thus  $U^1(\varphi) = U^2(\varphi)$  for  $\varphi \in C_b^6(\overline{\mathbb{R}})$ , which in turn implies that the solution  $U$  of (5.4) is unique (See Exercise 4.1 in [1]).  $\square$



The following Theorem 5.3 states that unique solution  $\{U\}$  of (5.4) is indeed the weak limit of any convergent subsequence of the measure-valued process  $\{U^n\}_n$ , in other words,  $\{U^n\}_n$  converges in distribution to  $\{U\}$ .

**Theorem 5.3.** *Under Assumption (A), any convergent subsequence of  $\{U^n\}_n$  has a limit  $U$  in distribution that is the unique  $\mathcal{D}_{\mathcal{M}_F(\overline{\mathbb{R}})}[0, \infty)$ -valued process  $U$  solving the following equation*

$$U_t(\varphi) = U_0(\varphi) + \int_0^t U_s(A\varphi)ds + \int_0^t U_s(h\varphi)dY_s + \Lambda_t^\varphi, \quad (5.8)$$

for  $\varphi \in C_b^6(\overline{\mathbb{R}})$ , where  $\Lambda_t^\varphi$  is defined as in (5.3).

*Proof.* From Proposition 5.3.20 in [15] and its extension to stochastic partial differential equation and infinitely dimensional stochastic differential equations, it follows that for solutions of stochastic partial differential equations, pathwise uniqueness implies uniqueness in law. This was done by Ondreját (see [8]) and Röckner, Schmuland and Zhang (see [21]).

Thus by Proposition 5.2 the solution  $U$  of (5.4) is unique in distribution.

Now let  $\{U^{n_k}\}_k$  be any convergent (in distribution) subsequence of  $\{U^n\}_n$  to a process  $U$ . We then verify that this process  $U$  solves (5.4), and then the uniqueness of solution of (5.4) implies that the original sequence  $\{U^n\}_n$  converges to  $U$  as well. Bearing in mind that  $U^{n_k}$  satisfies (3.9), it then essentially suffices to show that  $\Lambda_t^\varphi$  in (5.8), which is given by the weak limits of  $\sqrt{n}M_{[t/\delta]}^{n,\varphi}$  and  $\sqrt{n}B_t^{n,\varphi}$  in (3.9), does satisfy (5.3).

We first denote by

$$\bar{\Lambda}_t^\varphi \triangleq \Lambda_t^\varphi - \int_0^t \bar{\rho}_s^1(\Psi\varphi)ds$$

the martingale part of  $\Lambda_t^\varphi$ . Then we only need to show that  $\bar{\Lambda}^\varphi$  has the quadratic variation which is the same as that of  $\Lambda^\varphi$  in (5.3). In order to do so, we show that for all  $d, d' \geq 0$ ,  $0 \leq t_1 < t_2 < \dots < t_d \leq s \leq T$ ,  $0 \leq t'_1 < t'_2 < \dots < t'_{d'} \leq t \leq T$ , continuous bounded functions  $\alpha_1, \dots, \alpha_d$  on  $\mathcal{M}_F(\overline{\mathbb{R}})$  and continuous functions  $\alpha'_1, \dots, \alpha'_{d'}$  on  $\overline{\mathbb{R}}$ ; we have:

$$\tilde{\mathbb{E}} \left[ \left( \bar{\Lambda}_t^\varphi - \bar{\Lambda}_s^\varphi \right) \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] = 0, \quad (5.9)$$

and

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \left( \bar{\Lambda}_t^\varphi - \bar{\Lambda}_s^\varphi \right)^2 - \int_s^t \left\{ \bar{\rho}_r^2((\sigma\varphi')^2) + (\bar{\rho}_r^1(h\varphi'' - (h\varphi)'))^2 \right\} dr \right. \\ \left. - \sum_{i=[s/\delta]+1}^{[t/\delta]} (\rho_{i\delta}(\mathbf{1}))^2 \left[ \pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2 \right] \right] \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) = 0. \end{aligned} \quad (5.10)$$

To prove (5.9), we first observe the following:

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \xi_{[s/\delta]\delta}^n a_j^n(s) R_{s,j}^1(\varphi) ds \triangleq \lim_{n \rightarrow \infty} \Lambda_t^{n,R^1,\varphi} = \int_0^t \bar{\rho}_s^1(\Psi\varphi) ds, \quad (5.11)$$

the proof can be found in [19]. Then note that

$$\bar{\Lambda}_t^\varphi - \bar{\Lambda}_s^\varphi = U_t(\varphi) - U_s(\varphi) - \int_s^t U_r(A\varphi)dr - \int_s^t U_r(h\varphi)dY_r - \int_s^t \bar{\rho}_r^1(\Psi\varphi)dr,$$

thus showing (5.9) is equivalent to showing

$$\tilde{\mathbb{E}} \left[ \left( U_t(\varphi) - U_s(\varphi) - \int_s^t U_r(A\varphi)dr - \int_s^t U_r(h\varphi)dY_r - \int_s^t \tilde{\rho}_r^1(\Psi\varphi)dr \right) \times \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] = 0. \quad (5.12)$$

This equality will follow by virtue of the martingale property of  $\bar{\Lambda}_t^\varphi - \bar{\Lambda}_s^\varphi$ .

By virtue of the existence of  $\Lambda_T^n(\tilde{f}_k)$  in Lemma 4.3, it follows, for  $n' \in \mathbb{N}$ , that

$$\sup_{n'} \tilde{\mathbb{E}} \left[ (U^{n'}(\varphi))^2 \right] < \infty,$$

which implies that  $\{U^{n_k}\}$  is uniformly integrable (see II.20, Lemma 20.5 in [23]). Therefore we have that

$$\begin{aligned} \lim_{k \rightarrow \infty} \tilde{\mathbb{E}} \left[ U_t^{n_k}(\varphi) \prod_{i=1}^d \alpha_i(U_{t_i}^{n_k}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] &= \tilde{\mathbb{E}} \left[ U_t(\varphi) \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right], \\ \lim_{k \rightarrow \infty} \tilde{\mathbb{E}} \left[ U_s^{n_k}(\varphi) \prod_{i=1}^d \alpha_i(U_{t_i}^{n_k}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] &= \tilde{\mathbb{E}} \left[ U_s(\varphi) \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right]. \end{aligned}$$

By Burkholder-Davis-Gundy inequality, we know that

$$\sup_{n'} \tilde{\mathbb{E}} \left[ \left( \int_0^t U_r^{n'}(A\varphi)dr \right)^2 \right] < \infty;$$

thus we have

$$\lim_{k \rightarrow \infty} \tilde{\mathbb{E}} \left[ \int_s^t U_r^{n_k}(A\varphi)dr \prod_{i=1}^d \alpha_i(U_{t_i}^{n_k}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] = \tilde{\mathbb{E}} \left[ \int_s^t U_r(A\varphi)dr \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right].$$

Similarly, by Burkholder-Davis-Gundy inequality, we can show that

$$\sup_{n'} \tilde{\mathbb{E}} \left[ \left( \int_s^t U_r^{n'}(h\varphi)dY_r \right)^2 \right] < \infty,$$

we therefore have that (by Theorem 2.2 in [17]), since  $(U^{n_k}, Y)$  converges in distribution to  $(U, Y)$ , then  $(U^{n_k}, Y, \int_s^t U_r^{n_k}(h\varphi)dY_r)$  also converges in distribution to  $(U, Y, \int_s^t U_r(h\varphi)dY_r)$ , thus we have

$$\lim_{k \rightarrow \infty} \tilde{\mathbb{E}} \left[ \int_s^t U_r^{n_k}(h\varphi)dY_r \prod_{i=1}^d \alpha_i(U_{t_i}^{n_k}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] = \tilde{\mathbb{E}} \left[ \int_s^t U_r(h\varphi)dY_r \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right].$$

For  $\int_s^t \tilde{\rho}_r^1(\Psi\varphi)dr$ , we have

$$\lim_{k \rightarrow \infty} \tilde{\mathbb{E}} \left[ \Lambda_t^{n_k, R^1, \varphi} \prod_{i=1}^d \alpha_i(U_{t_i}^{n_k}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] = \tilde{\mathbb{E}} \left[ \int_s^t \tilde{\rho}_r^1(\Psi\varphi)dr \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right].$$

Now we have shown (5.12), and hence (5.9).

In order to show the second equality (5.10), we firstly make the following observations about the limits of the terms in (3.9):

- We have

$$\lim_{n \rightarrow \infty} \langle \sqrt{n} A^{n, \varphi} \rangle_t = \sum_{i=1}^{\lfloor t/\delta \rfloor} (\rho_{i\delta}(\mathbf{1}))^2 \left[ \pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2 \right]. \quad (5.13)$$

If we let

$$\bar{A}_t^\varphi \triangleq \sum_{i=1}^{\lfloor t/\delta \rfloor} \rho_{i\delta}(\mathbf{1}) \sqrt{\pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2} \Upsilon_i, \quad (5.14)$$

where  $\{\Upsilon_i\}_{i \in \mathbb{N}}$  is a sequence of independent identically distributed, standard normal random variables, and  $\left\{ \sqrt{\pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2} \Upsilon_i \right\}$  are mutually independent given the  $\sigma$ -algebra  $\mathcal{Y}$ ; then we have  $\langle \bar{A}^\varphi \rangle_t = \lim_{n \rightarrow \infty} \langle \sqrt{n} A^{n, \varphi} \rangle_t$ .

- For  $G_{\lfloor t/\delta \rfloor}^{n, \varphi}$ , we have

$$\lim_{n \rightarrow \infty} \left| \sqrt{n} G_{\lfloor t/\delta \rfloor}^{n, \varphi} \right| = 0 \quad \text{a.s.} \quad (5.15)$$

- We have

$$\lim_{n \rightarrow \infty} \left\langle \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\cdot \xi_{[s/\delta]\delta}^n a_j^n(s) R_{s,j}^2(\varphi) dY_s \right\rangle_t \triangleq \lim_{n \rightarrow \infty} \langle \Lambda^{n, R^2, \varphi} \rangle_t = \langle \Lambda^{R^2, \varphi} \rangle_t, \quad (5.16)$$

where

$$\Lambda_t^{R^2, \varphi} = c_\omega \int_0^t (\tilde{\rho}_s^1(h\varphi'' - (h\varphi)'')) dB_s^{(2)}, \quad (5.17)$$

$c_\omega$  is a constant and  $B^{(2)}$  is a Brownian motion independent of  $Y$ .

- We have that

$$\lim_{n \rightarrow \infty} \left\langle \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\cdot \xi_{[s/\delta]\delta}^n a_j^n(s) R_{s,j}^3(\varphi) dV_s^{(j)} \right\rangle_t \triangleq \lim_{n \rightarrow \infty} \langle \Lambda^{n, R^3, \varphi} \rangle_t = \langle \Lambda^{R^3, \varphi} \rangle_t, \quad (5.18)$$

where

$$\Lambda_t^{R^3, \varphi} = \int_0^t \sqrt{\tilde{\rho}_s^2((\sigma\varphi')^2)} dB_s^{(3)}, \quad (5.19)$$

$B^{(3)}$  is a Brownian motion independent of  $B^{(2)}$  and  $Y$ .

The proofs of these observations can be found in Appendix A.2.

From the above observations, we obtain that

$$\begin{aligned} & \tilde{\mathbb{E}} \left[ (\bar{\Lambda}_t^\varphi - \bar{\Lambda}_s^\varphi)^2 \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] \\ &= \lim_{k \rightarrow \infty} \tilde{\mathbb{E}} \left[ \left( (\langle \sqrt{n} A^{n_k, \varphi} \rangle_t - \langle \sqrt{n} A^{n_k, \varphi} \rangle_s) + \left( \langle \Lambda^{n_k, R^2, \varphi} \rangle_t - \langle \Lambda^{n_k, R^2, \varphi} \rangle_s \right) \right. \right. \\ & \quad \left. \left. + \left( \langle \Lambda^{n_k, R^3, \varphi} \rangle_t - \langle \Lambda^{n_k, R^3, \varphi} \rangle_s \right) \right) \times \prod_{i=1}^d \alpha_i(U_{t_i}^{n_k}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] \end{aligned}$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} \tilde{\mathbb{E}} \left[ \left( \sum_{i=[s/\delta]+1}^{[t/\delta]} (\rho_{i\delta}^{n_k}(\mathbf{1}))^2 \left[ \pi_{i\delta-}^{n_k}(\varphi^2) - (\pi_{i\delta-}^{n_k}(\varphi))^2 \right] \right. \right. \\
&\quad \left. \left. + \int_s^t (\tilde{\rho}_r^{n_k,1}(h\varphi'' - (h\varphi)''))^2 dr + \int_s^t \tilde{\rho}_r^{n_k,2}((\sigma\varphi')^2) dr \right) \times \prod_{i=1}^d \alpha_i(U_{t_i}^{n_k}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] \\
&= \tilde{\mathbb{E}} \left[ \left( \sum_{i=[s/\delta]+1}^{[t/\delta]} (\rho_{i\delta}(\mathbf{1}))^2 \left[ \pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2 \right] \right. \right. \\
&\quad \left. \left. + \int_s^t (\tilde{\rho}_r^1(h\varphi'' - (h\varphi)''))^2 dr + \int_s^t \tilde{\rho}_r^2((\sigma\varphi')^2) dr \right) \times \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right] \\
&= \tilde{\mathbb{E}} \left[ \left( \langle \bar{\Lambda} \cdot \varphi \rangle_t - \langle \bar{\Lambda} \cdot \varphi \rangle_s \right) \prod_{i=1}^d \alpha_i(U_{t_i}) \prod_{j=1}^{d'} \alpha'_j(Y_{t'_j}) \right]; \tag{5.20}
\end{aligned}$$

and (5.10) follows from this identity.  $\square$

**Corollary 5.4.** *Under Assumption (A), for and  $t \geq 0$  define  $\bar{U}_t^n \triangleq \sqrt{n}(\pi_t^n - \pi_t)$ . Then  $\{\bar{U}_t^n\}_n$  converges in distribution to a unique  $D_{\mathcal{M}_F(\bar{\mathbb{R}})}[0, \infty)$ -valued process  $\bar{U} = \{\bar{U}_t : t \geq 0\}$ , such that, for any test function  $\varphi \in C_b^6(\bar{\mathbb{R}})$ ,*

$$\bar{U}_t(\varphi) = \frac{1}{\rho_t(\mathbf{1})} (U_t(\varphi) - \pi_t(\varphi)U_t(\mathbf{1})), \tag{5.21}$$

where  $U$  satisfies (5.4).

*Proof.* By the fact that

$$\pi_t^n(\varphi) - \pi_t(\varphi) = \frac{1}{\rho_t(\mathbf{1})} (\rho_t^n(\varphi) - \rho_t(\varphi)) - \frac{\pi_t^n(\varphi)}{\rho_t(\mathbf{1})} (\rho_t^n(\mathbf{1}) - \rho_t(\mathbf{1})),$$

and  $\rho_t^n(\varphi) \rightarrow \rho_t(\varphi)$ , a.s. and  $\pi_t^n(\varphi) \rightarrow \pi_t(\varphi)$  a.s. (see Remark A.3), we have the result.  $\square$

**Remark 5.5.** *The central limit theorem in this paper is proven in Sections 4 and 5 with  $\varepsilon = 1/2$ . However, it should be noted that the result also holds when  $\varepsilon \in (0, 1/2)$ , and the corresponding proof is similar. Therefore the proof of the main result of the paper, Theorem 1.1, is completed without additional arguments for different  $\varepsilon$ .*

**Remark 5.6.** *In this chapter we view  $\{U^n\}_{n \in \mathbb{N}}$  and its weak limit  $\{U\}$  as processes with sample paths in  $D_{\mathcal{M}_F(\bar{\mathbb{R}})}[0, \infty)$ , which is complete and separable. In fact,  $U$  takes value in a smaller space  $\mathcal{M}_F(\bar{\mathbb{R}})$  (i.e.  $U$  is a  $D_{\mathcal{M}_F(\bar{\mathbb{R}})}[0, \infty)$ -valued random variable). In other words,  $U$  has no mass ‘escaping’ to infinity. This is shown by using the same approach as in Section 5 in [3].*

*Since the weak topology on  $\mathcal{M}_F(\bar{\mathbb{R}})$  coincides with the trace topology from  $\mathcal{M}_F(\bar{\mathbb{R}})$  to  $\mathcal{M}_F(\bar{\mathbb{R}})$ , it follows that  $U$  has sample paths in  $D_{\mathcal{M}_F(\bar{\mathbb{R}})}[0, \infty)$ . It then suffices to show that for arbitrary  $t$ , there exists a sequence of compact sets  $\{K_p\}_{p>0} \in \bar{\mathbb{R}}$  (possibly depending on  $t$ ) which exhaust  $\bar{\mathbb{R}}$  such that for all  $\varepsilon > 0$ ,*

$$\lim_{p \rightarrow \infty} \tilde{\mathbb{P}} \left[ \sup_{s \in [0, t]} \left( U_s(\mathbf{1}_{K_p^c}) \right) \geq \varepsilon \right] = 0, \tag{5.22}$$

where  $K_p^c$  denotes the complement of  $K_p$ . The proof of (5.22) can be found in Section 5 in [3].

## 6 Conclusions

In this paper, we analyse the Gaussian mixture approximations to the solution of the nonlinear filtering problem. In addition to the  $L^2$ -convergence result obtained in [5], we prove a central limit type theorem of the Gaussian mixture approximation, and find that the optimal value for the parameter  $\varepsilon$ , which measures the “Gaussianity” of the approximating system, is  $1/2$ . It can be seen that, asymptotically (as  $n \rightarrow \infty$ ), the mean square error between the approximating measure and the true solution of the filtering problem is (roughly) of order  $1/n$ , and the recalibrated error converges in distribution to a unique measure-valued process.

It should also be noted that the central limit theorem obtained in this paper is based on the approximating system under which the Multinomial branching algorithm is chosen. It is also worth studying the central limit theorem for the approximating system under the Tree Based Branching Algorithm, and this is left as future work.

## A Appendix

### A.1 Almost sure limits of $\pi^n$ and $\rho^n$

**Lemma A.1.** *If the approximation  $\pi^n$  is defined by (3.2), in other words,*

$$\pi_t^n(\varphi) = \sum_{j=1}^n \bar{a}_j^n(t) \int_{\mathbb{R}} \varphi \left( v_j^n(t) + y \sqrt{\omega_j^n(t)} \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} \right) dy;$$

*then we have*

$$\pi_t(\varphi) = \lim_{n \rightarrow \infty} \pi_t^n(\varphi) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \bar{a}_j^n(t) \varphi(v_j^n(t)). \quad (\text{A.1})$$

*That is, asymptotically, the variances of the Gaussian measures do not contribute to the approximation, and the combination of positions and weights provide a good approximation.*

*Proof.* See Appendix B in [19]. □

As a direct consequence, we have the following corollary for the unnormalised approximation  $\rho^n$ :

**Corollary A.2.** *If the approximation  $\rho^n$  is defined as*

$$\rho_t^n(\varphi) = \xi_t^n \pi_t^n(\varphi) = \xi_t^n \sum_{j=1}^n \bar{a}_j^n(t) \int_{\mathbb{R}} \varphi \left( v_j^n(t) + y \sqrt{\omega_j^n(t)} \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} \right) dy;$$

*then we have*

$$\rho_t(\varphi) = \lim_{n \rightarrow \infty} \rho_t^n(\varphi) = \lim_{n \rightarrow \infty} \xi_t^n \sum_{j=1}^n \bar{a}_j^n(t) \varphi(v_j^n(t)). \quad (\text{A.2})$$

**Remark A.3.** *By Lemma A.1 we know asymptotically as  $n \rightarrow \infty$ , the Gaussian mixture approximation performs just as good as the classic particle filters. Furthermore, from Chapter 8 in [1] and Lemma A.1, we know that*

$$\rho_t^n(\varphi) \rightarrow \rho_t(\varphi) \quad \text{and} \quad \pi_t^n(\varphi) \rightarrow \pi_t(\varphi) \quad \text{almost surely.}$$

## A.2 Proof of (5.13), (5.15), (5.16), and (5.18)

**Lemma A.4** ((5.13)). *Assume the conditions in Proposition 4.4 hold, then*

$$\lim_{n \rightarrow \infty} \langle \sqrt{n} A^{n,\varphi} \rangle_t = \sum_{i=1}^{\lfloor t/\delta \rfloor} (\rho_{i\delta}(\mathbf{1}))^2 \left[ \pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2 \right]. \quad (\text{A.3})$$

If we let

$$\bar{A}_t^\varphi \triangleq \sum_{i=1}^{\lfloor t/\delta \rfloor} \rho_{i\delta}(\mathbf{1}) \sqrt{\pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2} \Upsilon_i, \quad (\text{A.4})$$

where  $\{\Upsilon_i\}_{i \in \mathbb{N}}$  is a sequence of independent identically distributed, standard normal random variables, and  $\left\{ \sqrt{\pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2} \Upsilon_i \right\}_i$  are mutually independent given the  $\sigma$ -algebra  $\mathcal{Y}$ ; then we have  $\langle \bar{A}^\varphi \rangle_t = \lim_{n \rightarrow \infty} \langle \sqrt{n} A^{n,\varphi} \rangle_t$ .

*Proof.* Note that  $A^{n,\varphi}$  is a discrete time martingale, then

$$\begin{aligned} \lim_n \langle \sqrt{n} A^{n,\varphi} \rangle_t &= \lim_n \sum_{i=1}^{\lfloor t/\delta \rfloor} (\rho_{i\delta}^n(\mathbf{1}))^2 \left[ \sum_{j=1}^n \bar{a}_j^n(i\delta-) (\varphi(X_j^n(i\delta)))^2 - \left( \sum_{j=1}^n \bar{a}_j^n(i\delta-) \varphi(X_j^n(i\delta)) \right)^2 \right] \\ &= \sum_{i=1}^{\lfloor t/\delta \rfloor} (\rho_{i\delta}(\mathbf{1}))^2 \left[ \pi_{i\delta-}(\varphi^2) - (\pi_{i\delta-}(\varphi))^2 \right], \end{aligned}$$

here we made use of Lemma A.1 and Remark A.3.

The second part of the lemma is obvious.  $\square$

**Lemma A.5** ((5.15)). *Assume the conditions in Proposition 4.4 hold, then*

$$\lim_{n \rightarrow \infty} \left| \sqrt{n} G_{\lfloor t/\delta \rfloor}^{n,\varphi} \right| = 0 \quad a.s.. \quad (\text{A.5})$$

*Proof.* For  $G^{n,\varphi}$ , we know that

$$\sqrt{n} G_{\lfloor t/\delta \rfloor}^{n,\varphi} = \sum_{i=1}^{\lfloor t/\delta \rfloor} \sum_{j=1}^n \sqrt{n} \xi_{i\delta}^n \bar{a}_j^n(i\delta-) \left[ \varphi(X_j^n(i\delta)) - \tilde{\mathbb{E}}(\varphi(X_j^n(i\delta))) \right],$$

first note that  $X_j^n(i\delta) \sim N(v_j^n(i\delta), \omega_j^n(i\delta))$  and  $X_j^n$ s are mutually independent ( $j = 1, \dots, n$ ), also note the fact that  $\omega \sim \mathcal{O}(1/\sqrt{n})$ ; if we let  $Z_j^n(i\delta) \triangleq X_j^n(i\delta) - \tilde{\mathbb{E}}(X_j^n(i\delta))$  then  $Z_j^n(t) \sim \mathcal{N}(0, \omega_j^n(t))$ , and then by making use of the central moments of Gaussian random variables, we have

$$\begin{aligned} &\tilde{\mathbb{E}} \left[ \left( \sum_{i=1}^{\lfloor t/\delta \rfloor} \sum_{j=1}^n \sqrt{n} \xi_{i\delta}^n \bar{a}_j^n(i\delta-) \left[ \varphi(X_j^n(i\delta)) - \tilde{\mathbb{E}}(\varphi(X_j^n(i\delta))) \right] \right)^{12} \middle| \mathcal{Y}_{i\delta-} \right] \\ &\leq 2 \|\varphi'\|_{0,\infty}^{12} \tilde{\mathbb{E}} \left[ \left( \sum_{i=1}^{\lfloor t/\delta \rfloor} \sum_{j=1}^n \sqrt{n} \xi_{i\delta}^n \bar{a}_j^n(i\delta-) Z_j^n(i\delta) \right)^{12} \middle| \mathcal{Y}_{i\delta-} \right] \\ &\leq C^T \|\varphi\|_{1,\infty}^{12} \|\sigma\|_{0,\infty}^{12} \delta^6 n^9 \sum_{j=1}^n (\xi_{i\delta}^n \bar{a}_j^n(i\delta-))^{12}; \end{aligned}$$

then by taking the expectation on both sides, we have

$$\begin{aligned} \tilde{\mathbb{E}} \left[ \left( \sqrt{n} G_{[t/\delta]}^{n,\varphi} \right)^{12} \right] &\leq C^T \|\varphi\|_{1,\infty}^{12} \|\sigma\|_{0,\infty}^{12} \delta^6 n^9 \sum_{j=1}^n \tilde{\mathbb{E}} \left[ \left( \xi_{i\delta}^n \bar{a}_j^n(i\delta-) \right)^{12} \right] \\ &\leq C^T \|\varphi\|_{1,\infty}^{12} \|\sigma\|_{0,\infty}^{12} \delta^6 n^9 \sum_{j=1}^n \sqrt{\tilde{\mathbb{E}} \left[ (\xi_{i\delta}^n)^{24} \right] \tilde{\mathbb{E}} \left[ \left( \bar{a}_j^n(i\delta-) \right)^{24} \right]} \leq \frac{\beta_{\varphi,\sigma,\delta}^T}{n^2}, \end{aligned}$$

where

$$\beta_{\varphi,\sigma,\delta}^T = C^T \sqrt{c_1^{T,24} e^{c_{24}T}} \|\varphi\|_{1,\infty}^{12} \|\sigma\|_{0,\infty}^{12} \delta^6$$

is a constant independent of  $n$ . Then similar to the proof of Lemma A.1, we have the result.  $\square$

**Lemma A.6** ((5.16)). *Assume the conditions in Proposition 4.4 hold, then*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \xi_{[s/\delta]}^n a_j^n(s) R_{s,j}^1(\varphi) ds = \Lambda_t^{R^1, \varphi}, \quad (\text{A.6})$$

where

$$\Lambda_t^{R^1, \varphi} = c_\omega \int_0^t \tilde{\rho}_s^1(\Psi\varphi) ds; \quad (\text{A.7})$$

$c_\omega$  is a constant, and the operator  $\Psi$  is defined by

$$\Psi\varphi = \frac{f\varphi'''}{2} + \frac{\sigma\varphi^{(4)}}{4} - \frac{3(A\varphi)''}{2}.$$

*Proof.* Since

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \xi_{[s/\delta]}^n a_j^n(s) R_{s,j}^1(\varphi) ds \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^t \xi_{[s/\delta]}^n a_j^n(s) \left\{ \omega_j^n(s) \left[ \left( \frac{f\varphi'''}{2} + \frac{\sigma\varphi^{(4)}}{4} \right) (v_j^n(s)) - I_j(A\varphi) \right] \right\} ds \\ &= \lim_{n \rightarrow \infty} c_\omega \int_0^t \tilde{\rho}_s^{n,1}(\Psi\varphi) ds = c_\omega \int_0^t \tilde{\rho}_s^1(\Psi\varphi) ds, \end{aligned}$$

we have the required result.  $\square$

**Lemma A.7** ((5.18)). *Assume the conditions in Proposition 4.4 hold, then*

$$\lim_{n \rightarrow \infty} \left\langle \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\cdot \xi_{[s/\delta]}^n a_j^n(s) R_{s,j}^2(\varphi) dY_s \right\rangle_t = \left\langle \Lambda^{R^2, \varphi} \right\rangle_t, \quad (\text{A.8})$$

where

$$c_\omega \int_0^t (\tilde{\rho}_s^1(h\varphi'' - (h\varphi)'')) dB_s^{(2)}, \quad (\text{A.9})$$

$c_\omega$  is a constant and  $B^{(2)}$  is a Brownian motion independent of  $Y$ .



*Proof.* Observe that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left\langle \int_0^\cdot \frac{1}{\sqrt{n}} \sum_{j=1}^n \xi_{[s/\delta]\delta}^n a_j^n(s) R_{s,j}^2(\varphi) dY_s \right\rangle_t \\
&= \lim_{n \rightarrow \infty} \int_0^t \left( \frac{1}{2\sqrt{n}} \sum_{j=1}^n \xi_{[s/\delta]\delta}^n a_j^n(s) \omega_j^n(s) [(h\varphi'' - (h\varphi)'')(v_j^n(s))] \right)^2 ds \\
&= \lim_{n \rightarrow \infty} c_\omega^2 \int_0^t (\tilde{\rho}_s^{n,1} (h\varphi'' - (h\varphi)''))^2 ds = c_\omega^2 \int_0^t (\tilde{\rho}_s^1 (h\varphi'' - (h\varphi)''))^2 ds = \langle \Lambda^{R^2, \varphi} \rangle_t;
\end{aligned}$$

and then we have the result.  $\square$

**Lemma A.8.** *Assume the conditions in Proposition 4.4 hold, then*

$$\lim_{n \rightarrow \infty} \left\langle \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\cdot \xi_{[s/\delta]\delta}^n a_j^n(s) R_{s,j}^3(\varphi) dV_s^{(j)} \right\rangle_t = \langle \Lambda^{R^3, \varphi} \rangle_t, \quad (\text{A.10})$$

where

$$\Lambda_t^{R^3, \varphi} = \int_0^t \sqrt{\tilde{\rho}_s^2 ((\sigma\varphi')^2)} dB_s^{(3)},$$

$B^{(3)}$  is a Brownian motion independent of  $B^{(2)}$  and  $Y$ .

*Proof.* Note that  $\omega_j^n \propto \frac{1}{\sqrt{n}}$ , then by the same approach as in the proof of Lemma A.1 in [19], we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left\langle \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\cdot \xi_{[s/\delta]\delta}^n a_j^n(s) R_{s,j}^3(\varphi) dV_s^{(j)} \right\rangle_t \\
&= \lim_{n \rightarrow \infty} \int_0^t \tilde{\rho}_s^{n,2} ((\sigma\varphi')^2) ds = \int_0^t \tilde{\rho}_s^2 ((\sigma\varphi')^2) ds = \langle \Lambda^{R^3, \varphi} \rangle_t.
\end{aligned} \quad (\text{A.11})$$

We then have the result.  $\square$

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